

Stationary vortex flows and macroscopic Zeno effect in Bose-Einstein condensates with localized dissipation

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We theoretically demonstrate a possibility to observe the macroscopic Zeno effect in an effectively two-dimensional (pancake-shaped) repulsive Bose-Einstein condensate subjected to a strong narrow dissipation. We show that the dissipation can generate stable stationary nonlinear flows which bear either zero or non-zero topological charge (vorticity). The superfluid flows towards the dissipative defect compensate the atomic losses. The macroscopic Zeno effect manifests itself in a nonmonotonous dependence of the inward current density on the strength of the dissipation.

I. INTRODUCTION

Bose-Einstein condensates (BEC), as one of a few quantum phenomena directly observable at a macroscopic scale, allow for detailed study of the relation between micro and macro worlds. In the mean-field approximation, a BEC is described by the Gross-Pitaevskii equation (GPE) [1, 2] for the macroscopic wavefunction, which accounts for the external trap potential and for the inter-atomic interactions, the latter giving origin to the nonlinearity in the GPE. This is a conservative model which describes well many experimental setups if the losses are negligible.

In the meantime, in many practical situations, losses are naturally present and (or) cannot be avoided. In particular, inelastic light-atom interactions [1] and inelastic two- and three-body inter-atomic interactions [3] lead to linear and nonlinear dissipative terms in the meanfield description. Emergent evolution of a BEC in linear and nonlinear dissipative lattices was studied in [4] and [5], respectively.

One can consider condensates subjected to even stronger losses which, on the one hand cannot be treated as a perturbation, and, on the other hand, appear to be a useful tool for controlling and manipulating the system. Destruction of quantum states due to measurements, probing, or ionization by means of scattering are examples of sources of relatively strong losses. One of the consequences of the respective “dissipative” evolution of quantum states is their non-exponential decay, known since the pioneering works [6], and even total suppression of the decay under continuous measurement [7]. The latter phenomenon is known as the *quantum Zeno effect* (for a review see [8]). For the first time, the quantum Zeno effect was observed experimentally in vapors of Beryllium atoms [9]. In subsequent experimental studies, the phenomenon was observed by measuring escape of cold atoms loaded in an accelerating optical lattice [10], exploring atomic spin motion controlled by circularly polarized light [11], studying an externally driven mixture of two hyperfine states of a ^{87}Rb BEC [12], and inducing production of cold molecular gases in an optical lattice [13]. On the other hand, the quantum Zeno effect has also received considerable attention in theoretical works [14].

More recently, it has been suggested in [15] that macroscopic (phenomenological) analog of the Zeno effect can be addressed using a mean-field model of a BEC in a double-well trap with removal of atoms from one of the potential wells. Comparison of the quantum simulations with the respective GPEs with a dissipative term has shown very good agreement, which immediately suggested that the phenomenon can be observed in a wider class of physical systems, such as, for instance, optical waveguides with Kerr nonlinearity [16], whose dynamics is described by the equations fully analogous to the GPE with a double-well trap. Such a macroscopic manifestation of the phenomenon is referred to as the *macroscopic Zeno effect*.

The macroscopic Zeno effect is particularly well adjusted for some experimental setting, such as probing of a BEC with an electron beam [17, 18], where the condensed atoms are eliminated through the ionization resulting from electron-atom collisions. The macroscopic Zeno effect manifests itself in a nonmonotonic dependency of the atomic losses on the beam intensity (i.e. on the effective dissipation in terms of the mean-field approximation). Very recently, the macroscopic Zeno effect was observed experimentally [19].

The nonlinear dynamics in the presence of a localized defect is discussed already for many years; see e.g. [20] where interaction of kinks of the driven long Josephson junction with a dissipative defect in a form of Dirac delta was considered. Turning to the effect of localized dissipative defects on the evolution of a BEC reveals numerous striking phenomena. Among them we mention a possibility of manipulating the condensate (say performing switching and phase locking) [15] or excitation of different macroscopic patterns in the condensates [21]. It was also suggested that a strong enough dissipation can support quasi-stationary patterns [21] and stationary currents [22] which depend on the intensity and the width of the electron beam in a very nontrivial (nonmonotonic) way. In [23] it was shown that

the macroscopic Zeno effect can be observed not only in real space, but also in the Fourier one: a periodic dissipation (dissipative lattice) applied to a condensate allows for generating nondecaying Bloch states (see also [24]).

Most of the previous studies and, in particular, the theory of the macroscopic Zeno effect developed in [22], however, dealt with quasi-one-dimensional settings. Analysis of the phenomenon in the two-dimensional (2D) setting is the main goal of this work. More specifically, we will show that a narrow 2D dissipation can generate stable stationary nonlinear flows which bear either zero or non-zero topological charge (vorticity). Such modes exist due to compensation of the dissipative losses by the incoming flux (in this context we mention a recent for [25] where similar mechanism resulted in stationary 2D vortex light beams in a cubic-quintic medium with homogeneous nonlinear dissipation were reported). We also show that the current density of the stationary flows depends on the strength of the dissipation in the nonmonotonous way, which is a manifestation of the macroscopic Zeno effect.

II. THE MODEL

We describe the 2D macroscopic Zeno effect using a model which governs the macroscopic (meanfield) dynamics of a pancake-shaped BEC with a positive scattering length $a_s > 0$ (say, of ^{87}Rb atoms), subjected to a narrow electronic beam similar to the setting of [17, 19]. In particular, we consider the model where the beam radius is of order of $r_0 \approx 100$ nm while the radial dimension of the trap is $a_\perp \approx 3 \mu\text{m}$ ($\omega_\perp = 2\pi \times 500$ Hz). In this situation, $r_0/a_\perp \approx 1/30 \ll 1$ and one can explore the model of a narrow dissipative defect. To this end, we use the 2D GPE [15, 26]

$$i\frac{\partial\Psi}{\partial t} = -\nabla^2\Psi + g|\Psi|^2\Psi - i\Gamma(\mathbf{r})\Psi, \quad (1)$$

in the dimensionless variables $\mathbf{r} = (x, y)$, where $\hbar = 1$ and $m = 1/2$. We keep the nonlinear coefficient $g > 0$ as a free parameter for the sake of convenience; alternatively, it can be scaled out by rescaling the amplitude of the wavefunction. The function $\Gamma(\mathbf{r}) \geq 0$ describes effect of the external dissipation.

Generally speaking, the macroscopic Zeno effect is understood as a suppression of atomic losses subject to increasing strength of dissipation. If there is no pump of the atoms into condensate, then the BEC loses the atoms with the rate

$$\frac{dN}{dt} = -\int \Gamma(\mathbf{r})|\Psi(t, \mathbf{r})|^2 d\mathbf{r}, \quad (2)$$

where $N = \int |\Psi(t, \mathbf{r})|^2 d\mathbf{r}$ is the number of atoms in the BEC, and the macroscopic Zeno effect manifests itself in decrease of the decay rate as the intensity of the localized dissipation grows. In this statement, the effect was observed experimentally [19].

In our present study, we pose a slightly different problem. We consider an idealized situation when the dissipation, localized at $\mathbf{r} = 0$, is applied to an infinite sample of the BEC (i.e. number of atoms N is infinite). Then the dissipation induces a particle current directed from infinity towards the center, i.e. towards the dissipative defect. The density of the inward current can be adjusted to compensate exactly effect of the losses, which means that the condensate enters a stationary regime, i.e. the density distribution of the BEC and the density of the current do not depend on time t . In what follows, such structures are referred to as stationary flows. It is clear that the inward current density of the stationary flow, generally speaking, depends on the strength of the dissipation. Then the macroscopic Zeno effect, if exists, will manifest itself through a nonmonotonous dependence of the current density on the strength of dissipation. Such a statement of the problem is justified by the fact that the rate of atomic losses is proportional to the current density towards the defect necessary to compensate losses and to ensure stationary flows.

III. STATIONARY 2D FLOWS

In order to formalize the statement of the problem, we rewrite the GP-equations in the hydrodynamic form

$$\frac{1}{2}\frac{\partial n}{\partial t} + \nabla(n\mathbf{v}) + \Gamma(\mathbf{r})n = 0 \quad (3)$$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\mathbf{v}^2 + gn - \frac{\nabla^2\sqrt{n}}{\sqrt{n}}\right) = 0 \quad (4)$$

where $n(t, \mathbf{r}) = |\Psi(t, \mathbf{r})|^2$ is the local density, $\mathbf{v}(t, \mathbf{r}) = \nabla\Theta(t, \mathbf{r})$ is the superfluid velocity, and $\Theta(t, \mathbf{r}) = \arg \Psi(t, \mathbf{r})$ is the phase of the macroscopic wavefunction.

In what follows the consideration will be reduced to the radially symmetric dissipative defect, i.e. we consider $\Gamma(\mathbf{r}) \equiv \Gamma(r)$ (where $r = |\mathbf{r}|$). Moreover we consider dissipation having a finite support, i.e.

$$\Gamma(r) \equiv 0, \quad \text{for } r > \ell \quad (5)$$

and centered at $\mathbf{r} = 0$ (this form of the dissipative term models, in particular, the electronic beam used in [17–19]). Respectively, ℓ will be referred to as a width of the dissipation defect. Then the strength of the dissipation is proportional either to the maximal amplitude Γ_0 of the dissipation (in our case $\Gamma_0 = \Gamma(0)$) or to the width ℓ .

Radially symmetric dissipation allows us to look for two-dimensional radially symmetric stationary vortex solutions of GP equation (1) for which $n(t, \mathbf{r}) \equiv \rho^2(r)$ and $\Theta(t, \mathbf{r}) \equiv -\mu t + q\varphi + \theta(r)$, with φ being the azimuthal coordinate, the integer q characterizing the topological charge (vorticity) of the stationary flow, and μ being the chemical potential, i.e.

$$\Psi(t, \mathbf{r}) = \psi(r)e^{-i\mu t + iq\varphi}, \quad \psi(r) = \rho(r)e^{i\theta(r)}, \quad (6)$$

We also introduce the radial component of the superfluid velocity $v(r) = d\theta(r)/dr$ and note that the chemical potential is given as $\mu = g\rho_\infty^2$, where $\rho_\infty = \lim_{r \rightarrow \infty} \rho(r)$. Then substituting the expression (6) into the GPE (1), we obtain the following system of two equations:

$$\rho_{rr} + \frac{1}{r}\rho_r - \frac{j^2}{r^2\rho^3} + \left(\mu - \frac{m^2}{r^2}\right)\rho - g\rho^3 = 0, \quad (7)$$

$$j_r + r\Gamma(r)\rho^2 = 0, \quad (8)$$

where

$$j(r) = rv(r)n(r) \quad (9)$$

is the radial current density which is necessary to compensate the losses induced by the dissipative term $\Gamma(\mathbf{r})$.

We are interested in solutions (6) that correspond to stationary flows directed from infinity towards the center $r = 0$. We assume that the atomic density $n(r)$ is uniform at infinity, i.e. $n(r)$ approaches a fixed constant n_∞ as $r \rightarrow \infty$. Without loss of generality, in what follows we consider $n_\infty = 1$ (other values of n_∞ can be addressed by means of rescaling the nonlinear coefficient g). Since $\Gamma(r) \equiv 0$ for $r > \ell$, the density of the radial inward current $j(r)$ is constant for large r : $j(r) = j_\infty$ for $r \geq \ell$. Then for any stationary flow the chemical potential is fixed as $\mu = gn_\infty$.

For any stationary state, the current density j_∞ is such that the inward flow compensates exactly the effect of the losses $\Gamma(r)$. Hence we conclude that the macroscopic Zeno effect (if any) manifests itself in a nonmonotonous dependence of j_∞ on the amplitude Γ_0 or on the width ℓ of the dissipation.

Since $\Gamma(r)$ is zero outside of the bounded domain $r < \ell$, one finds that at $r \rightarrow \infty$ the limiting value of the density is approached algebraically (notice that in the 1D case the respective asymptotic is exponential [22]):

$$\rho(r) = \rho_\infty - \frac{1}{2gr^2} \left(\frac{m^2}{\rho_\infty} + \frac{j_\infty^2}{\rho_\infty^5} \right) - O\left(\frac{1}{r^4}\right), \quad (10)$$

where $\rho_\infty = \sqrt{n_\infty}$. Behavior of the radial velocity as $r \rightarrow \infty$ can be described as

$$v(r) = \frac{j_\infty}{r\rho^2(r)} = \frac{j_\infty}{\rho_\infty^2 r} + \frac{j_\infty}{\rho_\infty^3 gr^3} \left(\frac{m^2}{\rho_\infty} + \frac{j_\infty^2}{\rho_\infty^5} \right) + O\left(\frac{1}{r^5}\right). \quad (11)$$

Respectively, as $r \rightarrow \infty$, the behavior of argument $\theta(r)$ of the stationary wavefunction $\psi(r)$ is logarithmic in the leading order:

$$\theta(r) \approx \theta(R) + \frac{j_\infty}{\rho_\infty^2} \ln \frac{r}{R}, \quad (12)$$

where $r \geq R \gg \ell$.

Turning to behavior of the solutions in the vicinity of the origin, i.e. in the limit $r \rightarrow 0$, we show that if the topological charge is zero, i.e. if $q = 0$, then $\rho(0) \neq 0$ and $v(r) \sim r$. Indeed, let us make the most general assumption that in the leading order $\rho = \rho_0 r^\alpha$ and $v \sim r^{2\beta-1}$ where α and β are, so far, arbitrary constants. Then, from (8) in the leading order we obtain $r^{2(\alpha+\beta-1)} \sim \Gamma_0 r^{2\alpha}$, i.e. $\beta = 1$ and $v \sim r$. Substituting the obtained asymptotic for v in (7), and requiring α to be different from zero, we obtain in the leading order that the three terms in (7) are respectively of order of r^{-2} , r^4 and $(\rho_0^2 r^{2\alpha} - \rho_\infty^2)$. These terms cannot cancel each other in the limit $r \rightarrow 0$, i.e. the

assumption about nonzero α is contradictory, and in the leading order $\rho \approx \rho_0 \neq 0$ at $r \rightarrow 0$. For nonzero topological charge, $q \neq 0$, the asymptotic at $r \rightarrow 0$ reads $\rho(r) = r^q(\rho_0 + o(1))$, which is typical for vortex-like solutions.

We conclude this section by emphasizing that chosen dissipation is smooth but has a *finite* support $\ell > 0$. To understand that the width of the defect ℓ is a relevant parameter (for the 1D case this was shown in [22]) for the observation of the Zeno effect, let us consider a stationary solutions $n_t = 0$. Integrating (3) over a disc domain, of the radius ℓ , covering the defect region, we obtain

$$\int_{r < \ell} \nabla(n\mathbf{v})d^2\mathbf{r} = - \int_{r < \ell} \Gamma(r)n d^2\mathbf{r} \quad (13)$$

Now from the Green's divergence theorem we compute (recall the definition (9))

$$2\pi\ell n(\ell)v(\ell) = 2\pi j(\ell) = - \int_{r < \ell} \Gamma(r)n d^2\mathbf{r} \quad (14)$$

(recall that v is the radial component of the velocity and we consider radially symmetric solutions $n(r)$ and $v(r)$). Since we assumed that the dissipation has a finite support, Eq. (5), we have that $j(\ell) = j_\infty$. Thus, if $n(r)$ is not "too" singular function at $r = 0$, i.e. is bounded in the limit $r \rightarrow 0$, in the leading order we have that in the limit $\ell \rightarrow 0$

$$j_\infty \approx -\frac{n(\ell)}{2\pi} \int_{r < \ell} \Gamma(r)d^2\mathbf{r} \quad (15)$$

In other words $|j_\infty|$ is proportional to the integral strength of the dissipation $\int_{r < \ell} \Gamma(r)d^2\mathbf{r}$ and it monotonously grows with Γ_0 (or ℓ) at fixed ℓ (or Γ_0) meaning that in the limit $\ell \rightarrow 0$ one cannot observe the macroscopic Zeno effect. Formula (15) reveals also that $|j_\infty| \propto \Gamma_0$ at $\Gamma_0 \rightarrow 0$ and ℓ fixed, and $|j_\infty| \propto \ell^{2+2q}$ at $\ell \rightarrow 0$ and Γ_0 fixed. These types of behavior we will observe in Fig. 1 below (the upper and lower lines).

IV. NUMERICAL STUDY OF THE STATIONARY CURRENTS

For numerical study of stationary flows we choose the dissipation in the form

$$\Gamma(r) \equiv \begin{cases} \Gamma_0 (r^2/\ell^2 - 1)^2 & \text{for } r \leq \ell, \\ 0 & \text{for } r > \ell. \end{cases} \quad (16)$$

Then $\Gamma(r)$ is a smooth function with finite support.

Since experimental verification of the predicted effects would be only possible using stable stationary currents, for all the results reported below we tested the linear stability of the respective solution. To this end, we used the standard expression for the perturbed solution

$$\Psi = e^{-i\mu t + iq\varphi} \left[\psi(r) + a_\kappa(r)e^{i\omega t + iq\varphi} + b_\kappa^*(r)e^{-i\omega^* t - iq\varphi} \right], \quad (17)$$

where a_κ and b_κ characterize the small perturbation. After substituting this expression into (1) and subsequent linearization around a_κ and b_κ , one arrives at a set of decoupled eigenvalue problems

$$\begin{pmatrix} L_{+, \kappa} & -g\psi^2 \\ g(\psi^2)^* & -L_{-, \kappa} \end{pmatrix} \begin{pmatrix} a_\kappa \\ b_\kappa \end{pmatrix} = \omega \begin{pmatrix} a_\kappa \\ b_\kappa \end{pmatrix}, \quad \kappa = 0, 1, \dots, \quad (18)$$

with

$$L_{\pm, \kappa} = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + g\rho_\infty^2 \pm i\Gamma(r) - 2g|\psi|^2 - \frac{(\kappa \pm q)^2}{r^2}. \quad (19)$$

If for some $\kappa = 0, 1, \dots$ the spectrum of problem (18) contains an eigenvalue ω with negative imaginary part, then the corresponding stationary flow is unstable. Otherwise, the flow is linearly stable.

Passing now to numerical results, we considered stationary flows with zero vorticity ($q = 0$), as well as singly- ($q = 1$) and doubly-quantized ($q = 2$) vortex flows. We numerically identified stationary flows for different representative sets of the model parameters. The outcomes of our study are summarized in Fig. 1. In the upper row [Fig. 1(a)–(c)], we present dependencies of the absolute value of the radial current density $|j_\infty|$ on Γ_0 found for a fixed width of the dissipation ℓ and the strength of nonlinearity g (notice that j_∞ is always negative which follows from (8)). We observe

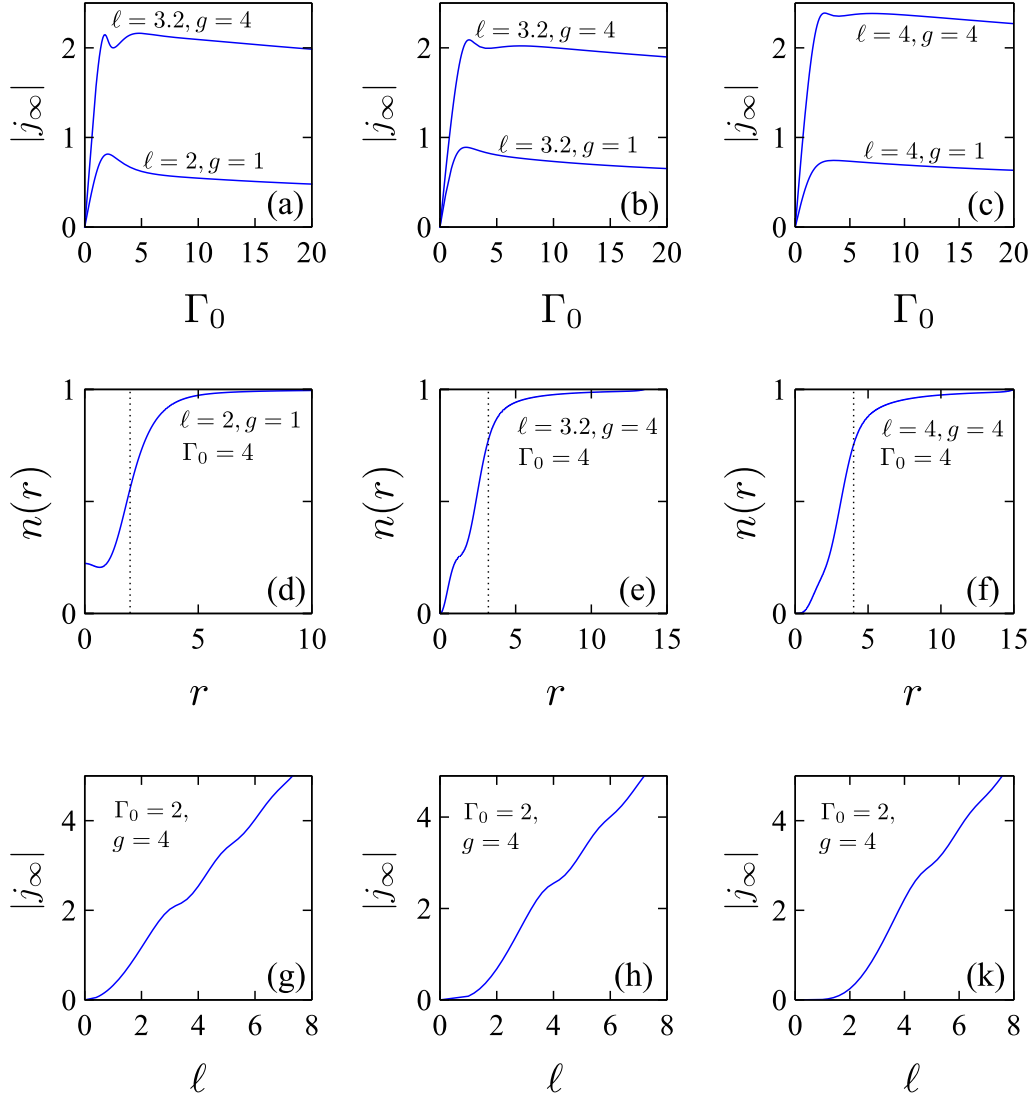


FIG. 1: (Color online) Dependencies $|j_\infty|$ vs. Γ_0 for $q = 0$ (a), $q = 1$ (b), $q = 2$ (c) obtained for fixed ℓ and g . Densities $n(r)$ of particular stationary flows from panels (a)–(c) are shown in (d)–(f). Vertical dotted lines correspond to the width of the dissipation, $r = \ell$. (g)–(k) Dependencies $|j_\infty|$ vs. ℓ for $q = 0$ (g), $q = 1$ (h), $q = 2$ (k) obtained for fixed Γ_0 and g . In all the panels, the specific values of parameters ℓ , Γ_0 and g are labelled in the plots (where applicable).

that the dependencies $j_\infty(\Gamma_0)$ are non-monotonous, i.e. they feature one (or several) local maxima, and the current density $|j_\infty|$ decreases for sufficiently large Γ_0 , which means that lower current density is necessary to compensate the effect of stronger dissipation. The observed nonmonotonicity is the evidence of the macroscopic Zeno effect. The effect is pronounced for the flows with either zero or nonzero vorticity q . Bearing in mind that the atomic losses in the dissipative domain are proportional to the obtained current densities, we observe remarkable qualitative similarity of the obtained curves with the experimentally observed ionization rate of condensed atoms subjected to the electronic beam (see Fig. 2 in [19]).

Typical spatial profiles of stationary density distribution $n(r)$ are shown in panels Fig. 1(d)–(f). It is interesting to notice that the density of a stationary flow with zero vorticity $q = 0$ [panel (d)] features a local maximum exactly at the maximum of the dissipation, i.e. at $r = 0$ (a similar effect was also observed in 1D case [22]).

We have also considered dependence of the current density j_∞ on the width of the defect ℓ for a given Γ_0 and g , Fig. 1(g)–(k). For all the considered cases, these dependencies were found to be monotonous, which contrasts the situation reported in [22] for the 1D case.

Using Eqs. (18), we examined linear stability of the flows shown on Fig. 1 by computing the eigenvalues ω for $\kappa = 0, 1, \dots, 5$. Our analysis indicated that the stationary flows shown on Fig. 1 are *stable*. We also confirmed stability of the presented stationary flows by means of computing their temporal evolution, which pointed out that

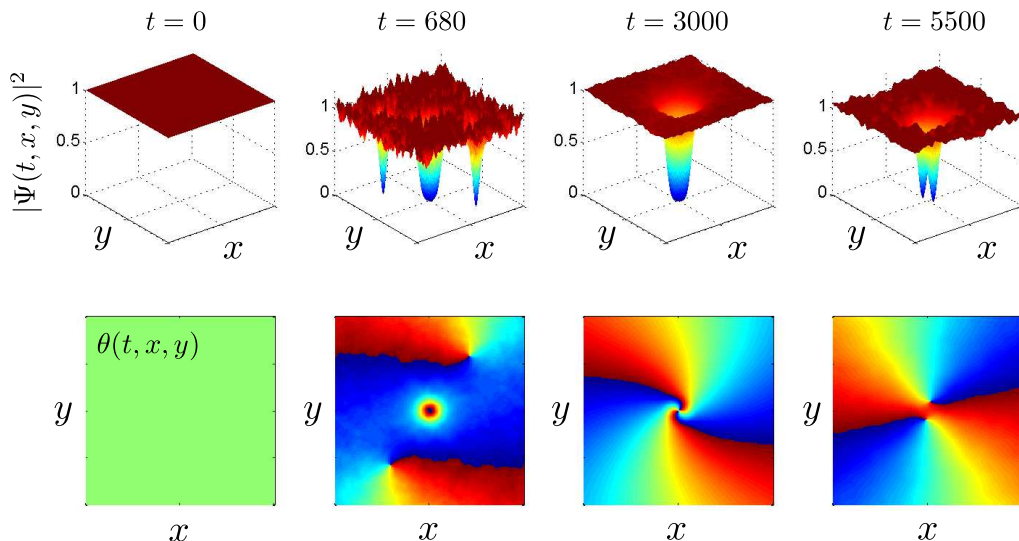


FIG. 2: (Color online) Formation (at $t \leq 3000$) and destruction (at $t > 3000$) of a doubly-quantized stationary flow. Parameters of the model: $\Gamma_0 = 5$ for $t \leq 3000$ ($\Gamma_0 = 0$ for $t > 3000$), $\ell = 4$, $g = 1$. Upper row shows density of the BEC $|\Psi(t, x, y)|^2$, while the lower row shows the phase $\theta = \arg \Psi$. All the panels are presented in the spatial window $(x, y) \in [-20, 20] \times [-20, 20]$.

the solutions are robust against relatively small perturbations introduced into the initial conditions. Moreover, we observed that the stationary flows feature attractor-like behavior, that is they can be obtained starting from a fairly general class of initial conditions. This fact is illustrated in Fig. 2 where the initial conditions with uniform density ($|\Psi(0, x, y)|^2 = 1$) undergo a sufficiently long transient process (see e.g. panels with $t = 680$) and eventually form a doubly-charged stationary flow ($t = 3000$). The resulting vorticity ($q = 2$ in Fig. 2) is determined by the boundary conditions which are chosen to “swirl” the condensate. The boundary conditions also fix the density of the condensate at the infinity ($|\Psi|^2 = 1$), but do not prescribe any specific value of the flux j_∞ incoming from the infinity. However, after the transient process the persistent incoming flux is established automatically, and the structure shown in Fig. 2 at $t = 3000$ is stationary and stable, i.e., it can persist for indefinitely long time, provided that the parameters of the model do not change. However, in the numerical experiment illustrated in Fig. 2, we present yet another dynamical scenario when, after an abrupt switching off the dissipation ($\Gamma_0 = 0$ for $t > 3000$), the obtained structure breaks into a pair of singly-charged vortices. In general, the GPE without dissipation [i.e., Eq. (1) with $\Gamma(\mathbf{r}) = 0$ which is tantamount to the defocusing nonlinear Schrödinger equation] is known to possess long-living doubly-charged vortex solutions [27]. In our case, however, the stationary density distribution at $t = 3000$ and $\Gamma_0 = 5$ is sufficiently different from that of the double vortex supported by the GPE without the dissipation so that no formation of the conservative doubly-charged occurs, but the density splits into two single-charged vortices.

V. CONCLUSION

To conclude, we have reported the existence of the macroscopic Zeno effect in a two-dimensional Bose-Einstein condensate subjected to a radially symmetric localized dissipation. The effect manifests itself in a non-monotonic dependence of the superfluid current density, necessary for supporting stable stationary flows, on the strength of the dissipation. When the latter is relatively small, its increase results in increase of the current density. However, after a certain threshold value further increase of the strength of the dissipation requires lower currents necessary to compensate the losses. The phenomenon is originated by a dissipation, which must have sufficiently strong localization (or even finite support, as in our case) but nonzero width (no Zeno effect can be observed for a point-like dissipation).

We have demonstrated that the effect can be observed for solutions that bear either zero or nonzero topological charge. In the latter case, the solutions represent stationary vortex flows. We found that such solutions are dynamically stable, either for single or multiple topological charge.

Our study was focused on an application to an atomic Bose-Einstein condensate. In the meantime, mathematical analogy allows one to predict a variety of other physical systems, such as optical waveguides, exciton-polariton and magnon condensates (see the discussion in [22]), where observation of macroscopic Zeno effect is also possible.

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